So far we have a formal way to describe how elements of a representation transform under an element of a group. But how can we build an invariant?

Given some thought, it might seem that combining two objects which transform "oppositely" would give an invariant. In fact this is exactly what we do!

We will take a cue from the familiar dot product, i.e. $\vec{v}\cdot\vec{w} = \#$ or $(v_1\ v_2\ v_3)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1w_1 + v_2w_2 + v_3w_3$

For any matrix representation r we can form the <u>dual representation</u> $\tilde{r}$ as follows:

If $A \in G$ then $r \to Ar$, $\tilde{r} \to (A^{-1})^T \tilde{r}$.

Then if we form $\tilde{r}^T r \to (A^{-1^T}\tilde{r})^T A r = \tilde{r}^T {A^{-1}}^{T^T} A r = \tilde{r}^T A^{-1} A r = \tilde{r}^T r$

In our example: $r = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$, $\tilde{r} = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$

If we choose $A = \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$ and $\begin{pmatrix}a\\b\\c\\d\end{pmatrix} \in r$ and $\begin{pmatrix}e\\f\\g\\h\end{pmatrix} \in \tilde{r}$ then:

$$\tilde{r}^T r = (e\ f\ g\ h)\begin{pmatrix}a\\b\\c\\d\end{pmatrix} \to \left[\underbrace{\begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}}_{A^{-1}}{}^T \begin{pmatrix}e\\f\\g\\h\end{pmatrix} \right]^T \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$$= ea + fb + gc + hd$$

$$= \left[\begin{pmatrix}h\\e\\f\\g\end{pmatrix}\right]^T \begin{pmatrix}d\\a\\b\\c\end{pmatrix} = (h\ e\ f\ g)\begin{pmatrix}d\\a\\b\\c\end{pmatrix} = hd + ea + fb + gc$$

same!!

Note: We can do this for complex representations as well, but since Lagrangians must be real, we only want real invariants so we form $\tilde{r}^* r$ instead.

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In the previous discussion I simply wrote down the dual representation. But we could ask if there is a systematic way to construct the dual $\tilde{r}$ if we are given r. For many cases this can be done if we are given a <u>metric</u>.

A metric g a map from an element of a representation r to a corresponding element of the dual representation $\tilde{r}$, i.e. $\tilde{r} = gr$. The metric will always be a symmetric matrix.

Based on this definition let's see what $\tilde{r}^T r =$ invariant implies about the metric g.

$$\tilde{r}^T r = (gr)^T r = \underbrace{r^T g^T r = r^T g r}_{\text{since } g \text{ is symmetric}} \longrightarrow (Ar)^T g Ar = r^T A^T g A r \quad \text{for some } A \in G$$

$$\overset{\text{then}}{=} r^T g r \quad \text{if} \quad \boxed{A^T g A = g}$$

The important lesson here is: If we have some representation r of a group G, then forming a dual representation $\tilde{r}$ with the metric g will give an invariant $\tilde{r}^T r$ if $A^T g A = g$ for $A \in G$.

We can turn this around to say: Given a representation r and a metric g, we can use the condition $A^T g A = g$ to find the transformations A which leave $\tilde{r}^T r$ invariant.

The latter statement is typically how we encounter symmetries in physics. We start with stuff (particles, fields, dynamical quantities, etc.) all of which form some representation. Then using some metric g, we can find a set of transformations that are symmetries of $\tilde{r}^T r$.

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As an example, consider vectors in 3D, $v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$ with metric $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

Then we can form dual vectors $\tilde{v} = gv$ and hence invariants $\tilde{v}^T v = v^T g v$
under any transformation A that satisfies $A^T g A = g$ or $\underbrace{A^T A = I}$ in this case.

This is the orthogonal condition.

In 3D, the A's would be 3x3 real matrices so the full set of transformations would be $O(3)$.

You might think that the A's in this case would be ordinary rotations in 3D, but we have to be careful.

Monday, February 06, 2012
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Rotations in 3D:

Form a compact, continuous, non-abelian group. We will denote rotations by R.

From our previous discussion we know $R^T R = I$ so $R \in O(3)$, but $O(3)$ contains more that just rotations.

Consider: $R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow R^T R = \mathbb{I}$

$R'_x(\theta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & \sin\theta \\ 0 & -\sin\theta & -\cos\theta \end{pmatrix} \Rightarrow R'^T R' = \mathbb{I}$ but this is <u>not</u> a rotation (we'll see what it is soon!)

How can we take $O(3)$ and pick out only rotations? Note: $\det R = +1$, $\det R' = -1$
So we can restrict to the elements of $O(3)$ that satisfy $\det = +1 \Rightarrow SO(3)$ special orthogonal group
But does this form a subgroup?

1. Closure $\quad A, B \in SO(3) \Rightarrow \det(AB) = \det A \det B = +1$
2. Identity $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$
3. Inverse $\quad R^T R = \mathbb{I} \Rightarrow R^{-1} = R^T$, but $\det(R^T R) = \det \mathbb{I} = +1 = \det R^T \underbrace{\cancel{\det R}}_{+1} \Rightarrow \det R^{-1} = +1$
4. Associativity (Matrix multiplication is naturally associative)

So what did we throw out? Essentially $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ reflection in x,y,z

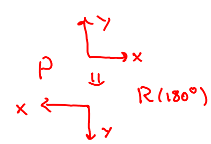
Note: $P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \underbrace{(P, \mathbb{I})}_{\mathbb{Z}_2}$ is a discrete subgroup of $O(3)$

To get any element of $O(3)$ we can start with some element of $SO(3)$ and combine it with P. Thus $\underbrace{O(3) = SO(3) \times \mathbb{Z}_2}_{\text{decomposition of } O(3) \text{ to subgroups}}$

Note: $O(3)$ with $\det = -1$ is <u>not</u> a subgroup!

The identity $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not part of it. ✗
Also $\det(AB) = \det A \det B = (-1)(-1) = +1$ ✗

Back to 2D: $\quad R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \det R = +1$
$\quad R(\theta) = \begin{pmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \Rightarrow \det R = +1$ $\Big\}$ both are in $SO(2)$!



What's the difference between 3D and 2D? In 2D, P which reflects x,y is just $R(180°) \in SO(2)$
In 3D, P which reflects x,y,z is <u>not</u> a rotation

We could consider $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \in O(2)$ with $\det R' = -1$. This can be associated with $P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
and used to decompose $O(2) = SO(2) \times \mathbb{Z}_2$

As another example, suppose we have a complex 2D representation $V = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ where v^1 and v^2 are complex numbers and we take the metric $g = I$. The $\tilde{V}^* V$ will be invariant under transformations by 2×2 complex matrices A provided $A^\dagger A = (A^T)^* A = I$.

The condition $A^\dagger A = I$ defines the unitary group $U(2)$.

Just as for $O(3)$, in order to restrict to continuous transformations we impose $\det A = +1$ and the have $SU(2)$ or the special unitary group in 2D. Clearly we can also have $SU(N)$.

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Counting continuous free parameters $\underbrace{\text{(or generators)}}_{\text{more on this later}}$ of a group

Example: SO(3)

$$R^T R = \mathbb{I} \Rightarrow \begin{pmatrix} \overbrace{a \quad b \quad c}^{9 \text{ parameters}} \\ d \quad e \quad f \\ g \quad h \quad i \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} a^2+b^2+c^2 = 1 \\ ad+be+cf=0 \\ ag+bh+ci=0 \\ da+be+fc=0 \\ d^2+e^2+f^2=1 \\ dg+eh+fi=0 \\ ga+hb+ic=0 \\ gd+he+if=0 \\ g^2+h^2+i^2=1 \end{array}\right\} \text{6 independent equations}$$

same: $ad+be+cf=0$ and $da+be+fc=0$
same: $ag+bh+ci=0$ and $ga+hb+ic=0$
same: $dg+eh+fi=0$ and $gd+he+if=0$

$\Rightarrow 9-6=3$ free parameters

What about $\det R = +1$?

$$\det R^T R = \det \mathbb{I} = +1$$
$$\det R^T \det R = +1$$
$$(\det R)^2 = +1$$
$$\underbrace{\det R = \pm 1}_{\text{2 real solutions}} \quad \text{due to } R^T R = \mathbb{I}$$

Generally: SO(N) has $\frac{1}{2}N(N-1)$ free parameter

SO(2) : 1

SO(3) : 3 (3 axes x, y, z or planes x-y, y-z, z-x of rotation).

SO(4) : 6 (6 planes of rotation x-y, y-z, z-x, x-w, y-w, z-w)

You might think this is special relativity but not so fast!

Wednesday, January 20, 2016
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If we take 4D vectors with $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ then the transformations Λ which satisfy $\Lambda^T g \Lambda = g$ and $\det \Lambda = +1$ form $SO(1,3)$ or the Lorentz group. We will develop this in much more detail in the next lectures.

Example: $SU(2)$

$\underbrace{8}_{}$ real parameters (4 complex numbers)

$U^{+}U = \mathbb{I} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^{+} & c^{+} \\ b^{+} & d^{+} \end{pmatrix} = \mathbb{I}$

$aa^{+} + bb^{+} = 1 \Rightarrow$ 1 equation since every term is real

Same $\left[\begin{array}{l} ac^{+} + bd^{+} = 0 \\ a^{+}c + b^{+}d = 0 \end{array}\right.$ $\Rightarrow$ 2 real equations since both real and imaginary parts are nonzero

$cc^{+} + dd^{+} = 1 \Rightarrow$ 1 equation since every term is real

In total we have 4 real independent equations: $8 - 4 = 4$ free parameters

What about $\det U = +1$? $\quad \det U^{+}U = \det \mathbb{I} = +1$

$\det U^{+} \det U = +1$

$(\det U^{T})^{*} \det U = +1$

$\underbrace{(\det U)^{*} \det U = +1}$

has continuous set of complex solutions $\det U = e^{i\theta}$

So $\det U = +1$ creates another nontrivial relation to be satisfied; $4 - 1 = 3$ independent free parameters

In general: $SU(N)$ has $N^2 - 1$ free parameters $[U(N)$ has $N^2]$

$U(1)$: 1 photon

$SU(2)$: 3 $W^{\pm}, Z^{0}$

$SU(3)$: 8 gluons

Well... sorta!